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# Systematic sampling with errors in sample locations

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SUMMARY

Systematic sampling of points in continuous space is widely used in microscopy and spatial surveys. Classical theory provides asymptotic expressions for the variance of estimators based on systematic sampling as the grid spacing decreases. However, the classical theory assumes that the sample grid is exactly periodic; real physical sampling procedures may introduce errors in the placement of the sample points. This paper studies the effect of errors in sample positioning on the variance of estimators in the case of one-dimensional systematic sampling. First we sketch a general approach to variance analysis using point process methods. We then analyze three different models for the error process, calculate exact expressions for the variances, and derive asymptotic variances. Errors in the placement of sample points can lead to substantial inflation of the variance, dampening of zitterbewegung, that is fluctuation effects, and a slower order of convergence. This suggests that the current practice in some areas of microscopy may be based on over-optimistic predictions of estimator accuracy.

*Some key words*: Asymptotic variance; Cavalieri estimator; Cumulative error; Moment measure; Perturbed systematic sampling; Point process; Spatial statistics; Stereology.

## 1. INTRODUCTION

Systematic sampling in continuous space is a useful technique in stereology, in ecological surveys and in other spatial sciences; see Baddeley & Jensen (2005) and references therein. In one dimension, a systematic sample is a grid of equally-spaced sample points, with fixed spacing

$t$, randomly shifted with respect to the origin. It may be constructed by setting $x_k = U + kt$ for all integers $k$, where $U$ is uniformly distributed on $[0, t)$. Systematic sampling can be used to estimate the integral

$$\Theta = \int_{\mathbb{R}} f(x)\,\mathrm{d}x$$

of any integrable function $f$, using the unbiased estimator $\hat{\Theta} = t \sum_k f(x_k)$.

Similarly, in two or three dimensions, a systematic sample is a randomly shifted regular grid of points with fixed geometry; the integral of any integrable function $f$ can be estimated by summing the function values at the sample points and multiplying by the area or volume of one tile in the grid. Such estimators were already known in the nineteenth and the early twentieth century (Delesse, 1847, 1848; Crofton, 1885; Rosiwal, 1898; Steinhaus, 1929, 1954; Thompson, 1930; Glagolev, 1933). Important early theoretical work on the performance of random grids and their relation to systematic sampling can be found in Moran (1966, 1968); see also Jones (1948).

A simple geometric example of systematic sampling in one dimension concerns the estimation of the volume of a bounded object in $\mathbb{R}^3$. Here, we may let $f(x)$ be the area of the intersection of the object with a horizontal plane at height $x \in \mathbb{R}$. The resulting sampling design is the egg-slicer design. The corresponding estimator is sometimes called the Cavalieri estimator, see Baddeley & Jensen (2005, p. 155), due to Cavalieri's principle, stating that two solid objects that have equal cross-sectional areas on all horizontal planes must have equal volumes. There are important applications of the Cavalieri estimator throughout biological science.

Systematic sampling, as formulated above, has experienced a renaissance in stereology since the mid-1980s. The main practical purpose of stereology is to estimate quantitative parameters of a spatial object from microscopical images of sections through the object. A very recent account of the mathematical and statistical foundations of stereology and the closely related field of stochastic geometry can be found in Weil & Schneider (2008).

Estimation of the precision of $\hat{\Theta}$ based on systematic sampling is a question of great current interest; see the recent volume of *Journal of Microscopy*, Mattfeldt (2006), devoted to this topic. There is extensive literature on the representation and approximation of the variance of $\hat{\Theta}$; see Baddeley & Jensen (2005, Ch. 13) and references therein. Matheron (1965, 1970) proposed studying this variance by means of the transitive theory, which provides a variance representation based on the Euler–MacLaurin formula; see also Cruz-Orive (1989). The variance can be expressed as the sum of the extension term, which gives the overall trend of the variance, the zitterbewegung or fluctuation term, which oscillates around zero, and higher-order terms. The extension term is used to estimate the variance of $\hat{\Theta}$. Matheron worked with the fundamental fact that the extension term depends on the behaviour of the geometric covariogram

$$g(z) = \int f(z + x) f(x) dx, \quad x \in \mathbb{R}$$

of $f$ near the origin. In Kiêu et al. (1999), a general form of the Euler–MacLaurin formula was derived, which reveals the connection between the variance of $\hat{\Theta}$ and the jumps of $f$ and its derivatives; see also Gual-Arnau & Cruz-Orive (1998). Two main findings of the classical theory are that, as the sample spacing decreases, the variance of $\hat{\Theta}$ decreases at a faster rate than under independent sampling, and that the variance does not decrease monotonically but fluctuates between high and low values, the zitterbewegung, because of resonance effects.

However, the classical theory assumes that the grid points are exactly periodic. In real sampling procedures, which may involve physically placing the sample points or physically cutting a material, the positions of the sample points may be subject to error. It appears to be unknown what effect these errors might have on the variance of $\hat{\Theta}$.

The key idea of the present paper is to describe the noisy sampling points by means of a point process $\Phi$. This approach has earlier been used with success in Pache et al. (1993) and Baddeley et al. (2006). The estimator to be considered is

$$\hat{\Theta} = \tau \sum_{x \in \Phi} f(x),$$

where $\tau > 0$ is a suitable normalization constant. The estimator $\hat{\Theta}$ will be denoted by a generalized Cavalieri estimator. We will study the case where the function $f$ is defined on the line. There are a number of important examples of this sampling situation in stereology, the most prominent ones being volume estimation from measurement of section areas, see Baddeley & Jensen (2005, p. 155) and references given above, and number estimation from disector counts; see Miller & Carlton (1895), Thompson (1932), Sterio (1984), and Cruz-Orive (1987).

We study three models for errors in sample locations. They are inspired by recent stereological studies of brain structure; see Dorph-Petersen (1999), Dorph-Petersen et al. (2005, 2007), and Sweet et al. (2005). The models are formulated here so that they have general probabilistic interest. In the first model, called perturbed systematic sampling, it is assumed that the sampling points are perturbed by independent and identically distributed errors $D_k$, $k \in \mathbb{Z}$. Under the second model, called systematic sampling with cumulative error, the increments between successive sampling points are independent and identically distributed. The last model, called systematic sampling with independent $p$-thinning, applies if observations are lost independently of each other with probability $p$. Perturbed systematic sampling and systematic sampling with independent $p$-thinning have earlier been discussed in another spatial sampling context in Lund & Rudemo (2000) under the names of displacement and thinning, respectively. One of the key results of this paper is that the effect of error in sample locations on the variance of the estimator $\hat{\Theta}$ may be substantial.

## 2. Preliminaries

First we mention some basics of point process theory needed in the sequel. For a detailed exposition, see Daley & Vere-Jones (2003, 2008) or Stoyan et al. (1995). Let $\mathcal{B}^d$ denote the Borel $\sigma$-algebra on $\mathbb{R}^d$. All point processes considered are assumed to be simple. Let $\Phi$ be a point process on $\mathbb{R}^d$ with intensity $m_1(x)$, for $x \in \mathbb{R}^d$. Then, $\Phi$ is said to be first-order stationary if $m_1(x)$ is constant, $m_1(x) = m$, say. The constant $m$ is called the intensity of the process. The process $\Phi$ is second-order stationary if it is first-order stationary and the density of the second-order factorial moment measure exists and is translation invariant, $m_{[2]}(x, y) = \tilde{m}_{[2]}(x - y)$. Here $\tilde{m}_{[2]}$ is the density of the second-order reduced factorial moment measure. Recall that $m_{[2]}(x, y)dxdy$ may be interpreted as the probability that two neighbourhoods of $x$ and $y$, respectively, each contains a point from the point process. The process $\Phi$ is strictly stationary if its distribution is invariant under translation. Let $\mathbb{1}_k$ denote the indicator of a set of K.

For a function $f : \mathbb{R}^d \to \mathbb{R}$, we define $\check{f} : \mathbb{R}^d \to \mathbb{R}$ by $\check{f}(x) = f(-x)$. The convolution of two functions $f, g : \mathbb{R}^d \to \mathbb{R}$ is denoted by $f * g$. Furthermore, we define the $k$-fold convolution of $f$ by $f^{k*} = f^{(k-1)*} * f$, $f^{1*} = f$ for $k = 2, 3, \ldots$. A function $f$ belongs to the space of locally integrable functions $L^1_{\mathrm{loc}}$, if for all compact sets $K$ the function $\mathbb{1}_K f$ is Lebesgue integrable. The space of integrable functions is denoted by $L^1$, while the space of essentially bounded functions is denoted by $L^{\infty}$. The space of $p$-times continuously differentiable functions is denoted by $C^p$. We write $C_0^p$ if they are also required to have compact support.

The following theorem (Baddeley et al., 2006), allows us to study the first- and second-order properties of estimators based on systematic sampling with errors. Let $f$ be a measurement function, that is, an integrable function with bounded support on $\mathbb{R}$. Define $\Theta = \int_{\mathbb{R}} f(x)dx$.

THEOREM 1. *Suppose that $\Phi$ is a first-order stationary point process with intensity $m_1(x) = m$, where $m > 0$. Then the generalized Cavalieri estimator $\hat{\Theta} = \tau \sum_{x\in\Phi} f(x)$ with $\tau = m^{-1}$ is an unbiased estimator of $\Theta$. If $\Phi$ is second-order stationary with density $\tilde{m}_{[2]}$ of the second-order reduced factorial moment measure, then*

$$\mathrm{var}(\hat{\Theta}) = \frac{g(0)}{m} + \frac{1}{m^2}\int_{\mathbb{R}} g(z)\tilde{m}_{[2]}(z)dz - \int_{\mathbb{R}} g(z)dz,$$

*where $g(z) = \int_{\mathbb{R}} f(z+x)f(x)dx$ is the geometric covariogram of $f$.*

The proof of Theorem 1 is based on standard techniques from point process theory.

## 3. MODELS FOR $\Phi$

### 3·1. *Perturbed systematic sampling*

Perturbed systematic sampling was briefly considered in Baddeley et al. (2006). We assume that the intended equally spaced sampling points $x_k = U + kt$ are perturbed by random errors $(D_k)_{k\in\mathbb{Z}}$, so that the actual locations are $y_k = x_k + D_k$. The random variable $U$ is uniformly distributed on $[0, t)$, where $t > 0$ is the intended spacing of the sampling points. The sequence $(D_k)_{k\in\mathbb{Z}}$ is independent and identically distributed with common density function $h$, which has bounded support. In relation to cutting of tissue in stereological studies, perturbed systematic sampling will, for example, be a reasonable model for devices consisting of an array of cutting blades; see Gundersen et al. (1988, Fig. 7). Slight drift of the blades while cutting will perturb the actual cut around the fixed position of each blade.

In Baddeley et al. (2006) the following representation of the variance is obtained:

$$\mathrm{var}(\hat{\Theta}) = tg(0) + t\sum_{n\in\mathbb{Z},\ n\neq 0} g * \check{h} * h(nt) - \int_{\mathbb{R}} g(z)dz. \tag{1}$$

The convolution $h * \check{h}$ is the density of $D_k - D_l$ for $k \neq l$.

### 3·2. *Systematic sampling with cumulative error*

In this model we assume that the actual locations $(y_k)_{k\in\mathbb{Z}}$ of the sampling points are such that the increments $w_k = y_k - y_{k-1}$, $k \in \mathbb{Z}$, are independent and identically distributed with density $h : \mathbb{R}_+ \to \mathbb{R}_+$ and finite expectation $t > 0$. We choose the starting distribution $\tilde{H}$ for $y_1$ as $\tilde{H}(x) = t^{-1}\int_0^x \{1 - H(y)\}\,dy$, where $H$ is the distribution function of $h$. Applying Daley & Vere-Jones (2008, Theorem 13.3.I), we then have that $\Phi = (y_k)_{k\in\mathbb{Z}}$ is a strictly stationary point process with finite intensity $m = t^{-1}$. Systematic sampling with cumulative error is appropriate if the sampling procedure works like a meat slicer, where each successive section is cut by advancing the material towards a stop plate a fixed distance from the slicing blades. If the block advance is slightly variable, e.g. due to elasticity of the material leading to a variable degree of compression, then we get cumulative errors.

LEMMA 1. *Let $\Phi$ be a point process that follows the systematic sampling with cumulative error model with increment density $h$ with mean $t > 0$. Then, $\Phi = (y_k)_{k\in\mathbb{Z}}$ is second-order stationary* *with intensity equal to $t$ and the second-order reduced factorial moment measure has density*

$$\tilde{m}_{[2]}(x) = \frac{1}{t}\sum_{k=1}^{\infty}\{h^{k*}(x) + \check{h}^{k*}(x)\},$$

*where $h^{k*}$ denotes the $k$-fold convolution of $h$. The density $\tilde{m}_{[2]}$ is locally integrable.*

The proof of Lemma 1 can be found in the Appendix. Lemma 1 and Theorem 1 yield

$$\mathrm{var}(\hat{\Theta}) = tg(0) + t\int_{\mathbb{R}}\sum_{k=1}^{\infty} g(z)\{h^{k*}(z) + \check{h}^{k*}(z)\}dz - \int_{\mathbb{R}} g(z)dz. \tag{2}$$

### 3·3. *Systematic sampling with independent $p$-thinning*

Suppose we have sampling points at locations $\Psi = (y_k)_{k\in\mathbb{Z}}$, which form a second-order stationary point process, the so-called centre process, with intensity $m = t^{-1}$ and second-order reduced factorial density $\tilde{m}^c_{[2]}$. Let $p > 0$ be the probability that the value of $f$ cannot be determined at location $y_k$. Let $(U_k)_{k\in\mathbb{Z}}$ be a sequence of independent and identically distributed uniform random variables on [0, 1] and independent of $(y_k)_{k\in\mathbb{Z}}$. The resulting point process is $\Phi = \{y_k : U_k > p\}$. The intensity of $\Phi$ is $(1-p)m$ and the second-order reduced factorial moment measure of $\Phi$ has density $(1-p)^2\tilde{m}^c_{[2]}(y)$. Therefore, we obtain

$$\mathrm{var}(\hat{\Theta}) = \frac{g(0)}{(1-p)m} + \frac{1}{m^2}\int_{\mathbb{R}} g(z)\tilde{m}^c_{[2]}(z)\,dz - \int_{\mathbb{R}} g(z)dz. \tag{3}$$

Only the first term on the right-hand side is different from the formula for the variance of the generalized Cavalieri estimator based on the process $\Psi$; see Theorem 1.

## 4. Limiting behaviour

### 4·1. *Perturbed systematic sampling*

In this section we study the asymptotic behaviour of the variance for perturbed systematic sampling. The measurement function will from now on be denoted by $f$, the density of the error distribution by $h$. For perturbed systematic sampling, we can rewrite (1) as

$$\begin{aligned}\mathrm{var}(\hat{\Theta}) &= tg(0) + t\sum_{n\in\mathbb{Z},\, n\neq 0} g*\check{h}*h(nt) - \int_{\mathbb{R}} g(z)dz\\ &= t\{g(0) - g*\check{h}*h(0)\} + t\sum_{n\in\mathbb{Z}} g*\check{h}*h(nt) - \int_{\mathbb{R}} g*\check{h}*h(z)dz,\end{aligned}$$

where by Fubini's theorem it is easy to see that $\int_{\mathbb{R}} g*\check{h}*h(z)dz = \int_{\mathbb{R}} g(z)dz$. Recall that the geometric covariogram $g$ is defined as $g(z) = \int_{\mathbb{R}} f(x)f(x+z)dx$, where $f$ is the measurement function. Using this definition, it is easy to check that $g*\check{h}*h(z) = \int_{\mathbb{R}} f*h(x+z)f*h(x)dx$. Define $F = f*h$. We now consider $F$ as the measurement function. Its covariogram is $G(z) = g*\check{h}*h(z)$. Let $\hat{W} = t\sum_{j\in\mathbb{Z}} F(U+jt)$, where $U$ is uniformly distributed on $[0,t)$. We then obtain

$$\mathrm{var}(\hat{W}) = t\sum_{j\in\mathbb{Z}} G(jt) - \int_{\mathbb{R}} G(z)dz; \tag{4}$$

see Gual-Arnau & Cruz-Orive (1998), Baddeley & Jensen (2005, Ch. 13.18) and references therein. As we want to study the asymptotic behaviour of the variance of $\hat{\Theta}$ as $t \to 0$, we need to specify how the error density $h$ depends on $t$. Throughout this section we assume that

$$h_t(x) = \frac{1}{t} h_0 \left( \frac{x}{t} \right), \quad x \in \mathbb{R} \quad (t > 0),$$

where $h_0$ is a probability density function belonging to the class $\mathcal{C}_K$ of Lebesgue measurable functions with compact support and a finite number of jumps of finite size. In this model the standard deviation of the error depends linearly on $t$. Models where the standard deviation of the error relative to $t$ increases when $t$ decreases would also be interesting. The need for such models may arise in practice if a cutting device with a fixed error is used over a wide range of different spacings $t$. We have yet to investigate the effect of such different models for $h_t$ on the asymptotic behaviour of the variance.

For a function $q : \mathbb{R} \to \mathbb{R}$ let

$$s_q(x) = \lim_{y \to x^+} q(y) - \lim_{y \to x^-} q(y), \quad x \in \mathbb{R},$$

where we assume that the limits are defined everywhere. Let $D_q = \text{supp}(s_q)$. The function $q$ is said to be $(m, p)$-piecewise smooth, for $m, p = 0, 1, \ldots,$ if $q^{(l)} \in \mathcal{C}_K$ for all $l = 0, \ldots, m + p$ and $D_{q^{(l)}} = \emptyset$ for $0 \leqslant l < m$. Thus, an $(m, p)$-piecewise smooth function has compact support. Furthermore, all its derivatives of order less than $m$ are continuous while derivatives of order $m$ up to $m + p$ have a finite number of jumps of finite size.

PROPOSITION 1. *Let $f$ be an $(m, 1)$-piecewise smooth measurement function. Then its covariogram $g$ is $(2m + 1, 1)$-piecewise smooth and the variance of the generalized Cavalieri estimator has the following expansion as $t \to 0$:*

$$\begin{aligned} \text{var}(\hat{\Theta}) = {} & t\{g(0) - g * \check{h}_t * h_t(0)\} \\ & - t^{2m+2} s_{g^{(2m+1)}}(0) \int_{\mathbb{R}} h_0 * \check{h}_0(x) P_{2m+2}(x) dx + o(t^{2m+2}), \end{aligned} \tag{5}$$

*where $P_i(\cdot)$ denotes the $i$th Bernoulli polynomial. Let $c_2 = \int_{\mathbb{R}} x^2 h_0 * \check{h}_0(x) dx$. If $\text{supp}(h_0) \subseteq [-1/2, 1/2]$ and $m = 0$, (5) simplifies to*

$$\text{var}(\hat{\Theta}) = -t^2 \left( \frac{c_2}{2} + \frac{1}{12} \right) s_{g'}(0) + o(t^2). \tag{6}$$

*For $m \geqslant 1$, we have*

$$\text{var}(\hat{\Theta}) = -t^3 \frac{c_2}{2} g^{(2)}(0) + o(t^3). \tag{7}$$

The proof of Proposition 1 may be found in the Appendix.

*Remark* 1. The properties of piecewise smooth functions are studied in a set of unpublished lecture notes by Kien Kiêu entitled Three Lectures on Systematic Geometric Sampling, which appeared as *Memoirs* at the Department of Theoretical Statistics at the University of Aarhus in 1997. By Corollary 5.8 of these notes $s_{g^{(2m+1)}}(0) \neq 0$. From the definition of $g$ it is clear that for $m \geqslant 1$ we have $g^{(2)}(0) \neq 0$.

*Remark* 2. We define the Bernoulli polynomials as in Knopp (1996, Paragraph 297). For $x \in [0, 1]$ we first define inductively $\tilde{P}_0(x) = 1$, $\tilde{P}_1(x) = x - 1/2$ and for $i = 2, 3, \ldots$ let $\tilde{P}'_{i+1} = \tilde{P}_i$ and $\tilde{P}_i(0) = (1/n!)B_n$, where $B_n$ is the $n$th Bernoulli number. Then let

$P_i(x) = \tilde{P}_i(x - [x])$. So $P_i$ is bounded, 1-periodic and $P'_{i+1} = P_i$ for $i = 0, 1, \ldots,$ in particular $P_2(x) = (1/2)\{(x - [x])(x - [x] - 1) + 1/6\}$. In Kiêu's notes the Bernoulli polynomial $P_i$ is denoted by $P_{i,1}$.

### 4·2. *Systematic sampling with cumulative error*

We assume that the increment density for a certain spacing $t > 0$ is given by $h_t(x) = t^{-1}h_0(x/t)$, where $h_0$ is a probability density on the positive half-line with expected value 1. Define $u_t^+ = \sum_{k=1}^{\infty} h_t^{k*}$, $u_t^- = \sum_{k=1}^{\infty} \check{h}_t^{k*}$. The function $u_t^+$ is supported on the positive half-line, while $u_t^-$ is supported on the negative half-line. Furthermore, $u_t^{\pm}(x) = t^{-1}u_0^{\pm}(x/t)$. Rewriting (2) with this notation yields

$$\operatorname{var}(\hat{\Theta}) = tg(0) + \int_0^{\infty} g(z)u_0^+\left(\frac{z}{t}\right)dz + \int_{-\infty}^{0} g(z)u_0^-\left(\frac{z}{t}\right)dz - \int_{\mathbb{R}} g(z)dz. \tag{8}$$

The function $u_0^+$ is the renewal density of a renewal process with holding times that are independent and identically distributed with density $h_0$. The following theorem (Alsmeyer, 1991, para. 3.3.1, para. 13.2.2), reveals the asymptotic behaviour of $u_0^{\pm}$.

THEOREM 2. *Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a probability density with expectation $\mu > 0$ and let $u = \sum_{k=1}^{\infty} h^{k*}$. Suppose that $h \in L^{\infty}$ and $\lim_{s\to\infty} h(s) = 0$. Then*:

(a) *$u \in L^{\infty}$ and $u - h$ is continuous and bounded*;
(b) *$\lim_{s\to\infty} u(s) = \mu^{-1}$, where $\infty^{-1} = 0$*;
(c) *if $h$ is absolutely continuous and there is an integer $m \geqslant 2$, such that $\int_{\mathbb{R}} |x^{m-1}h'(x)|dx < \infty$ and the $m$th moment of $h$ exists, then $u(s) - \mu^{-1} = o(s^{1-m})$ as $s \to \infty$.*

We assume now that $h_0 \in L^{\infty}$ and that $\lim_{s\to\infty} h_0(s) = 0$. By part (b) of Theorem 2, we obtain for each $z \in [0, \infty)$ that $\lim_{t\to 0} g(z)u_0^+(z/t) = g(z)$ and analogously for $z \in (-\infty, 0]$ and $u_0^-$. Furthermore, $|g(z)u_0^+(z/t)| \leqslant \|u_0^+\|_{\infty}|g(z)| \in L^1$, using part (a) of Theorem 2, and again analogously for $u_0^-$. Lebesgue's dominated convergence theorem now implies

$$\lim_{t\to 0}\left\{\int_0^{\infty} g(z)u_0^+\left(\frac{z}{t}\right)dz + \int_{-\infty}^{0} g(z)u_0^-\left(\frac{z}{t}\right)dz\right\} = \int_{\mathbb{R}} g(z)dz$$

and hence $\lim_{t\to 0}\operatorname{var}(\hat{\Theta}) = 0$. The order of convergence is determined in the proposition below. If $h_0$ does not have expected value 1, then $\operatorname{var}(\hat{\Theta})$ does not converge to zero for $t \to 0$.

PROPOSITION 2. *Assume that $h_0$ satisfies the conditions in part (c) of Theorem 2 for some $m \geqslant 3$ and that the covariogram $g$ is continuous at 0 and bounded. Then the variance of the generalized Cavalieri estimator under the model of systematic sampling with cumulative error satisfies $\operatorname{var}(\hat{\Theta}) = tg(0)v^2 + o(t)$ as $t \to 0$, where $v^2 < \infty$ is the variance of a random variable with density $h_0$.*

The proof of Proposition 2 may be found in the Appendix.

### 4·3. *Systematic sampling with independent p-thinning*

PROPOSITION 3. *Let $f$ be an $(m, 1)$-piecewise smooth measurement function. Then its covariogram $g$ is $(2m + 1, 1)$-piecewise smooth and the variance of the generalized Cavalieri estimator under perturbed systematic sampling combined with independent $p$-thinning with thinning* *probability* $p > 0$ *satisfies*

$$\mathrm{var}(\hat{\Theta}) = t\frac{p}{1-p}g(0) + o(t), \quad t \to 0.$$

*Proof*. This follows by combining (3) with Proposition 1. □

PROPOSITION 4. *Assume that the conditions on* $h_0$ *of part* (*c*) *of Theorem* 2 *are fulfilled for some* $m \geqslant 3$ *and that the covariogram* $g$ *is continuous at* 0 *and bounded. Then the variance of the generalized Cavalieri estimator under systematic sampling with cumulative error combined with independent* $p$*-thinning with thinning probability* $p > 0$ *has the expansion*

$$\mathrm{var}(\hat{\Theta}) = tg(0)\left(\nu^2 + \frac{p}{1-p}\right) + o(t), \quad t \to 0,$$

*where* $\nu^2 < \infty$ *is the variance of a random variable with probability density* $h_0$.

*Proof*. This follows by combining (3) with Proposition 2. □

## 5. EXAMPLE

As an example, we have investigated the effect of errors in sample locations of section planes on the precision of the estimator of the volume of the unit ball. In this case, the measurement function $f$ and the geometric covariogram $g$ can be calculated explicitly; see Baddeley et al. (2006). The measurement function $f$ is $(1, \infty)$-piecewise smooth.

Figure 1(a) shows the variance of the generalized Cavalieri estimator under the model of perturbed systematic sampling. We chose a truncated normal distribution with mean zero for the error distribution. For a comparison, we also plotted the variance of the estimator under exact systematic sampling. The order of magnitude of the variance of the error distribution has been chosen in accordance with what has been found in recent morphological studies where the model of perturbed systematic sampling is appropriate; see Dorph-Petersen et al. (2005, 2007). The leading term of the asymptotic expansion of the variance as given by (7) can be seen as a straight line in the log-log scale used in the graph.

The variance of the generalized Cavalieri estimator under the model of systematic sampling with cumulative error is displayed in Fig. 1(b). The increment distribution is a truncated normal distribution with mean 1, variance $\sigma^2$ and truncation points 0 and 2. For the calculation, we approximated the $k$th fold convolution of the truncated normal density $h_0$ by a truncated normal density with mean $k$, truncation points 0 and $2k$ and variance $k^{1/2}\sigma^2$.

As shown in Fig. 1(b), cumulative error may have a substantial effect on the variance. For example, if 100 sections are used, exact sampling gives a very small coefficient of variation of about 0·002%. But for systematic sampling with cumulative error even with the smaller standard deviation of $\sigma = 0{\cdot}05$, the coefficient of variation is about 0·55%. The effect for perturbed systematic sampling, on the other hand, appears to be less significant.

## 6. DISCUSSION

The reason why random sampling experiments have become so important in biological applications of stereological methods is that most biological structures are highly organized and spatially inhomogeneous, so that sampling inference cannot be drawn from a single arbitrarily positioned sample; see Weibel (1978). A first mention of a design-based approach in stereology can be found in the far-sighted paper Thompson (1932); see also the accompanying paper

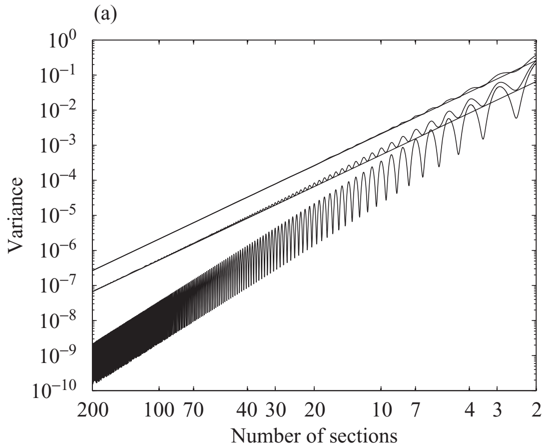


Fig. 1. Variance of the Cavalieri estimator of volume of a unit ball as a function of the expected number of sections $n$, shown on a log-log scale. Note that $t = 2/n$ as we are cutting a unit ball. The lower curves are based on exact systematic sampling. (a) the upper and middle curves were calculated using the model of perturbed systematic sampling with a truncated normal error distribution $h_0$ with mean zero, truncation points $\pm 1/2$ and standard deviation $\sigma = 0{\cdot}05$ (middle curve) and $\sigma = 0{\cdot}10$ (upper curve), respectively. The straight lines represent the main terms of the asymptotic expansion of the variances. (b) the upper and middle curves were calculated using the model of systematic sampling with cumulative error with a truncated normal increment distribution $h_0$ with mean 1, truncation points 0 and 2 and standard deviation $\sigma = 0{\cdot}05$ (middle) and $\sigma = 0{\cdot}10$ (upper), respectively. The straight lines represent the main terms of the asymptotic expansion of the variances.

Thompson et al. (1932) and Royall (1970). An alternative to randomization of sampling points would be to develop a stochastic model for the biological structure under study. This is, however, not needed for estimating parameters $\Theta$ expressible as integrals.

In the present paper, we have assessed the asymptotic variance of $\hat{\Theta}$ as $t \to 0$ in the case of systematic sampling with errors. It is remarkable that for all perturbation mechanisms presented, the zitterbewegung effect is asymptotically negligible as $t \to 0$. The example in § 5 shows that errors in the placement of sampling points may lead to a substantial inflation of the estimator variance.

There are a number of ways in which the methods presented here may be extended. Measurement functions $f$ with first-order derivative being noncontinuous with infinite jumps are not covered by the asymptotic theory for perturbed systematic sampling developed in the present paper. In the case of the classical Cavalieri estimator, the asymptotic variance has been derived for such measurement functions in García-Fiñana & Cruz-Orive (2000, 2004) and García-Fiñana (2006). The variance exhibits a fractional trend. The trend is often of order $t^{2p+2}$, typically with $0 < p < 1$. Another obvious extension concerns the effect on the variance of errors in placement of sampling points in the case where sampling in two or three dimensions is performed. Such errors in two-dimensional systematic sampling occur, for example, when subsampling tissue slabs for electron microscopy. A perforated sampling grid is randomly positioned over each tissue slab, and using a trocar, punch biopsies are obtained from each hole overlying the region of interest. Slight random error in the position of the punch is inevitable; see, for example, Tang & Nyengaard (2004, Fig. 12.1).

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## APPENDIX

### *Proofs*

In this Appendix, we refer to a set of lecture notes by Kien Kiêu entitled Three Lectures on Systematic Geometric Sampling, which appeared as *Memoirs* at the Department of Theoretical Statistics at the University of Aarhus in 1997. Below, these lecture notes are referred to as KK (1997).

*Proof of Lemma* 1. Denote by $\overset{\circ}{M}_1$ the first moment measure of the Palm distribution $\mathcal{P}_0$ of $\Phi$. We have

$$\overset{\circ}{M}_1(A) - \delta_0(A) = \mathrm{E}_{\mathcal{P}_0}\{\Phi(A\backslash\{0\})\}, \quad A \in \mathcal{B}.$$

Define $y_0' = 0$, $y_k' = \sum_{i=1}^{k} w_i$ and $y_{-k}' = \sum_{i=0}^{k-1} -w_{-i}$ for $k = 1, 2, \ldots$. We obtain

$$\mathrm{E}_{\mathcal{P}_0}\{\Phi(A\backslash\{0\})\} = \mathrm{E}\left\{\sum_{k=1}^{\infty} \mathbb{1}_A(y_k') + \sum_{k=1}^{\infty} \mathbb{1}_A(y_{-k}')\right\} = \int_A \sum_{i=1}^{\infty}\{h^{i*}(x) + \check{h}^{i*}(x)\}\, dx.$$

The term $\sum_{k=1}^{\infty} h^{k*}(x)$ is the renewal density of a renewal process with holding times that are independent and identically distributed with density $h$. Standard renewal theory yields that $\sum_{k=1}^{\infty} h^{k*}(x)$ is locally integrable, so in particular the series converges for almost all $x \in \mathbb{R}$; see, for example, Daley & Vere-Jones (2003, Ch. 4). The same argument holds for $\sum_{k=1}^{\infty} \check{h}^{k*}(x)$, where we have to consider a renewal process with reversed time. Therefore, $\tilde{m}_{[2]}(x) \in L^1_{loc}$, which implies the existence of the first moment measure $\overset{\circ}{M}_1$ of the Palm distribution. By Daley & Vere-Jones (2008, Proposition 13.2.VI) this implies the existence of the second-order reduced factorial moment measure $\tilde{M}_{[2]}(A) = m\{\overset{\circ}{M}_1(A) - \delta_0(A)\}$. Inserting $m = t^{-1}$ yields the claim. □

*Proof of Proposition* 1. Suppose that the measurement function $f$ is $(m, p)$-piecewise smooth with $p \geqslant 1$. Let $F_t = f * h_t$. Then, it follows from KK (1997, Proposition 5.6 and Corollary 5.8) that the covariogram $G_t = g * h_t * \check{h}_t$ of $F_t$ is $(2m+2)$-times continuously differentiable and

$$(g * h_t * \check{h}_t)^{(2m+2)} = g^{(2m+2)} * h_t * \check{h}_t + s_{g^{(2m+1)}} * h_t * \check{h}_t, \tag{A1}$$

where $s * q(x) = \sum_a s(a)q(x-a)$ for a function $s$ with finite support and a function $q$ whose support has nonzero Lebesgue measure. In KK (1997, Proposition 4.2) a refined Euler–MacLaurin formula for $(m, 1)$-piecewise smooth functions is given. The proof relies on a partial integration formula for piecewise smooth functions. We would like to apply the formula to the right-hand side of (4) with $G = G_t$. This is not directly possible as the error term approximations are only valid, if $G$ does not depend on $t$, but following the proof of KK (1997, Proposition 4.2), we obtain, using (A1),

$$\begin{aligned} \mathrm{var}(\hat{W}_t) = &-t^{2m+2}\int_{\mathbb{R}} g^{(2m+2)} * h_t * \check{h}_t(x) P_{2m+2}\left(\frac{x}{t}\right) dx \\ &- t^{2m+2} \sum_{a \in D_{g^{(2m+1)}}} s_{g^{(2m+1)}}(a) \int_{\mathbb{R}} h_t * \check{h}_t(x-a) P_{2m+2}\left(\frac{x}{t}\right) dx. \end{aligned} \tag{A2}$$

By KK (1997, Corollary 5.8), we always have $s_{g^{(2m+1)}}(0) \neq 0$ and, as $g$ is an even function, we obtain $s_{g^{(2m+1)}}(0) = 2g^{(2m+1)}(0^+)$, where $g^{(2m+1)}(0^+) = \lim_{x \to 0^+} g^{(2m+1)}(x)$.

The second term on the right-hand side of (A2) can be decomposed as

$$\begin{aligned} t^{2m+2} \sum_{a \in D_{g^{(2m+1)}}} s_{g^{(2m+1)}}(a) \int_{\mathbb{R}} h_t * \check{h}_t(x-a) P_{2m+2}\left(\frac{x}{t}\right) dx = t^{2m+2} s_{g^{(2m+1)}}(0) \int_{\mathbb{R}} h_0 * \check{h}_0(x) P_{2m+2}(x) dx \\ + t^{2m+2} \sum_{a \in D_{g^{(2m+1)}}, a \neq 0} s_{g^{(2m+1)}}(a) \int_{\mathbb{R}} h_0 * \check{h}_0\left(x - \frac{a}{t}\right) P_{2m+2}(x) dx. \end{aligned} \tag{A3}$$

For all $a \neq 0$ and $x \in \mathbb{R}$ the term $h_0 * \check{h}_0(x - a/t)P_{2m+2}(x)$ converges to zero as $t \to 0$. As $h_0 * \check{h}_0$ is compactly supported and bounded, Lebesgue's dominated convergence theorem shows that

$\int_{\mathbb{R}} h_0 * \check{h}_0\{x-(a/t)\}P_{2m+2}(x)dx$ converges to zero. Therefore, the second term of the right-hand side of (A3) converges to zero with order $o(t^{2m+2})$.

The asymptotic behaviour of the first term on the right-hand side of (A2) can be determined by the following reasoning. Suppose $g^{(2m+2)}$ is $(0,1)$-piecewise smooth. This is a stronger assumption than $g^{(2m+2)} \in \mathcal{C}_K$, which we obtain from the $(m,1)$-piecewise smoothness of $f$. Then we can again apply KK (1997, Proposition 5.6) to obtain $(g^{(2m+2)} * h_t * \check{h}_t)^{(1)} = g^{(2m+3)} * h_t * \check{h}_t + s_{g^{(2m+2)}} * h_t * \check{h}_t$. This derivative is again continuous, so partial integration of the integral in the first term on the right-hand side of (A2) yields

$$\int_{\mathbb{R}} g^{(2m+2)} * h_t * \check{h}_t(x)P_{2m+2}\left(\frac{x}{t}\right)dx = -t\int_{\mathbb{R}} g^{(2m+3)} * h_t * \check{h}_t(x)P_{2m+3}\left(\frac{x}{t}\right)dx$$
$$-t\sum_{a\in D_{g^{(2m+2)}}} s_{g^{(2m+2)}}(a)\int_{\mathbb{R}} h_t * \check{h}_t(x-a)P_{2m+3}\left(\frac{x}{t}\right)dx. \quad \text{(A4)}$$

Here we use that $\int P_{2m+3}(x/t)dx = t^{-1}P_{2m+2}(x/t)$.

It is not difficult to see that both terms on the right-hand side of (A4) converge with order of at least $O(t)$, hence the first term on the right-hand side of (A2) converges to zero with order $o(t^{2m+2})$. If instead of assuming that $g^{(2m+2)}$ is $(0,1)$-piecewise smooth, we only require that $g^{(2m+2)} \in \mathcal{C}_K$, as assumed in Proposition 1, then $g^{(2m+2)}$ is Riemann integrable, so for each $\varepsilon > 0$ there exists a step function $\tilde{g} \in \mathcal{C}_K$, such that $\tilde{g} \leqslant g^{(2m+2)}$ and

$$0 \leqslant \int_{\mathbb{R}} g^{(2m+2)}(x)dx - \int_{\mathbb{R}} \tilde{g}(x)dx \leqslant \frac{\varepsilon}{2\|P_{2m+2}\|_\infty}.$$

It is not difficult to check that this implies

$$0 \leqslant \int_{\mathbb{R}} g^{(2m+2)} * h_t * \check{h}_t(x)dx - \int_{\mathbb{R}} \tilde{g} * h_t * \check{h}_t(x)dx \leqslant \frac{\varepsilon}{2\|P_{2m+2}\|_\infty}$$

and

$$\left|\int_{\mathbb{R}} g^{(2m+2)} * h_t * \check{h}_t(x)P_{2m+2}\left(\frac{x}{t}\right)dx - \int_{\mathbb{R}} \tilde{g} * h_t * \check{h}_t(x)P_{2m+2}\left(\frac{x}{t}\right)dx\right| \leqslant \frac{\varepsilon}{2}.$$

As $\tilde{g}$ is $(0,1)$-piecewise smooth, we can apply the same argument as above in order to show that $\tilde{I} = \int_{\mathbb{R}} \tilde{g} * h_t * \check{h}_t(x)P_{2m+2}(x/t)dx = O(t)$. In particular $|\tilde{I}| \leqslant \varepsilon/2$, for $t$ small enough. This implies that

$$\left|\int_{\mathbb{R}} g^{(2m+2)} * h_t * \check{h}_t(x)P_{2m+2}\left(\frac{x}{t}\right)dx\right| \leqslant \varepsilon$$

for $t$ small enough, hence this integral tends to zero as $t \to 0$. Therefore, the first term on the right-hand side of (A2) converges to zero with order $o(t^{2m+2})$.

It only remains to show the order of convergence of the first term in (5). It is easy to see that $g(0) - g * \check{h}_t * h_t(0) = \int_{\mathbb{R}}\{g(0) - g(tx)\}\check{h}_0 * h_0(x)\,dx$. Fix $x \in \mathbb{R}$. For $t > 0$ small enough, we can use Taylor expansion to obtain

$$g(xt) - g(0) = \sum_{k=1}^{m} \frac{1}{(2k)!}g^{(2k)}(0)x^{2k}t^{2k} + \frac{1}{(2m+1)!}g^{(2m+1)}(\xi)x^{2m+1}t^{2m+1}$$

as all uneven continuous derivatives of $g$ are odd functions so they are zero at zero; $\xi$ is between 0 and $xt$. If $m = 0$ and $x > 0$, then $t^{-1}\{g(xt) - g(0)\} = g'(\xi)x \to g'(0^+)x$ as $t \to 0^+$. Using Lebesgue's dominated convergence theorem and $s_{g'}(0) = 2g'(0^+)$, one can deduce that $t\{g(0) - g * \check{h}_t * h_t(0)\} \sim -t^2\{s_{g'}(0)/2\} \int_{\mathbb{R}} |x| h_0 * \check{h}_0(x)dx$. If $\mathrm{supp}(h_0) \subseteq [-1/2, 1/2]$, one obtains (6) because

$$\begin{aligned}\mathrm{var}(\hat{\Theta}) &= -t^2 s_{g'}(0)\left\{\frac{1}{2}\int_{\mathbb{R}} |x| h_0 * \check{h}_0(x)dx + \int_{\mathbb{R}} P_2(x) h_0 * \check{h}_0(x)dx\right\} + o(t^2)\\ &= -t^2\left(\frac{c_2}{2} + \frac{1}{12}\right)s_{g'}(0) + o(t^2),\end{aligned}$$

using (5) and the definition of the second Bernoulli polynomial $P_2$. If $m \geqslant 1$, we obtain $t\{g(0) - g * \check{h}_t * h_t(0)\} = -t^3(c_2/2)g^{(2)}(0) + o(t^3)$ using dominated convergence and the boundedness of $g^{(2m+1)}$, hence (7) follows immediately from (5). □

*Proof of Proposition* 2. The assumptions on $h_0$ yield that $(u_0^+ - 1)$ is integrable. Using substitution, we obtain

$$\int_0^\infty g(z)\left\{u_0^+\left(\frac{z}{t}\right) - 1\right\}dz = t\int_0^\infty \{g(tz) - g(0)\}\{u_0^+(z) - 1\}dz + tg(0)\int_0^\infty \{u_0^+(z) - 1\}dz.$$

The first term on the right-hand side of the above equation converges with order $o(t)$ as $t \to 0$. This can be seen by using dominated convergence and the continuity of $g$ at 0. As $g$ is symmetric and $u_0^-(z) = u_0^+(-z)$, we obtain

$$\mathrm{var}(\hat{\Theta}) = tg(0)\left[2\int_0^\infty \{u_0^+(z) - 1\}dz + 1\right] + o(t), \qquad \text{(A5)}$$

using (8). Let $U$ be the renewal measure of the renewal process with holding times that are independent and identically distributed with density $h_0$. Then $u_0^+$ is a density for $U - \delta_0$. The function $\{u_0^+(z) - 1\}\mathbb{1}_{[0,K]}(z)$ converges in $L^1$ to $u_0^+(z) - 1$ as $K \to \infty$, therefore

$$\int_0^\infty \{u_0^+(z) - 1\}dz = \lim_{K\to\infty}\int \{u_0^+(z) - 1\}\mathbb{1}_{[0,K]}(z)dz = \lim_{K\to\infty}\{(U - \delta_0)([0, K]) - K\} = \frac{\nu^2 - 1}{2},$$

by Alsmeyer (1991, Theorem 3.4.1). Combining this with (A5) yields the claim. □

## References

Alsmeyer, G. (1991). *Erneuerungstheorie*. Stuttgart: B. G. Teubner.

Baddeley, A., Dorph-Petersen, K. A. & Jensen, E. B. V. (2006). A note on the stereological implications of irregular spacing of sections. *J. Microsc.* **222**, 177–81.

Baddeley, A. & Jensen, E. B. V. (2005). *Stereology for Statisticians*. Boca Raton, FL: Chapman & Hall/CRC.

Crofton, M. W. (1885). *Probability*. Encyclopaedia Britannica, 9th ed. London: Encyclopaedia Britannica, Inc.

Cruz-Orive, L. M. (1987). Particle number can be estimated using a disector of unknown thickness: the selector. *J. Microsc.* **145**, 121–42.

Cruz-Orive, L. M. (1989). On the precision of systematic sampling: a review of Matheron's transitive methods. *J. Microsc.* **153**, 315–33.

Daley, D. J. & Vere-Jones, D. (2003). *An Introduction to the Theory of Point Processes*, vol. I, 2nd ed. New York: Springer.

Daley, D. J. & Vere-Jones, D. (2008). *An Introduction to the Theory of Point Processes*, vol. II, 2nd ed. New York: Springer.

Delesse, A. (1847). Procédé mécanique pour déterminer la composition des roches. *C. R. Acad. Sci. Paris* **25**, 544–45.

Delesse, A. (1848). Procédé mécanique pour déterminer la composition des roches. *Ann. Mines* **13**, 379–88.

Dorph-Petersen, K.-A. (1999). Stereological estimation using vertical sections in a complex tissue. *J. Microsc.* **195**, 79–86.

Dorph-Petersen, K.-A., Pierri, J. N., Perel, J. M., Sun, Z., Sampson, A. R. & Lewis, D. A. (2005). The influence of chronic exposure to antipsychotic medications on brain size before and after tissue fixation: a comparison of haloperidol and olanzapine in macaque monkeys. *Neuropsychopharmacol.* **30**, 1649–61.

Dorph-Petersen, K.-A., Pierri, J. N., Wu, Q., Sampson, A. R. & Lewis, D. A. (2007). Primary visual cortex volume and total neuron number are reduced in schizophrenia. *J. Comp. Neurol.* **501**, 290–301.

García-Fiñana, M. (2006). Confidence intervals in Cavalieri sampling. *J. Microsc.* **222**, 146–57.

García-Fiñana, M. & Cruz-Orive, L. M. (2000). Fractional trend of the variance in Cavalieri sampling. *Image Anal. Stereol.* **19**, 71–79.

García-Fiñana, M. & Cruz-Orive, L. M. (2004). Improved variance prediction for systematic sampling on $\mathbb{R}$. *Statist.* **38**, 243–72.

Glagolev, A. A. (1933). On geometrical methods of quantitative mineralogic analysis of rocks. *Trans. Inst. Econ. Mining* **59**, 1–47.

Gual-Arnau, X. & Cruz-Orive, L. M. (1998). Variance prediction under systematic sampling with geometric probes. *Adv. Appl. Prob. (SGSA)* **30**, 889–903.

Gundersen, H. J. G., Bagger, P., Bendtsen, T. F., Evans, S., Korbo, L., Marcussen, N., Møller, A., Nielsen, K., Nyengaard, J. R., Pakkenberg, B., Sørensen, F. B., Vesterby, A. & West, M. J. (1988). The new stereological tools: disector, fractionator, nucleator and point sampled intercepts and their use in pathological research and diagnosis. *APMIS* **96**, 857–81.

Jones, A. E. (1948). Systematic sampling of continuous parameter populations. *Biometrika* **35**, 283–96.

Kiêu, K., Souchet, S. & Istas, J. (1999). Precision of systematic sampling and transitive methods. *J. Statist. Plan. Infer.* **77**, 263–279.

Knopp, K. (1996). *Theorie und Anwendung der unendlichen Reihen*. Berlin: Springer.

Lund, J. & Rudemo, M. (2000). Models for point processes observed with noise. *Biometrika* **87**, 235–49.

Matheron, G. (1965). *Les Variables Régionalisées et Leur Estimation*. Paris: Masson et Cie.

Matheron, G. (1970). *The Theory of Regionalized Variables and Its Applications*. Les Cahiers du Centre de Morphologie Mathématique de Fontainebleau 5. Fontainebleau, France: Ecole Nationale Supérieure des Mines de Paris, Fontainebleau.

Mattfeldt, T. (2006). Special volume on variance estimation in stereology. *J. Microsc.* **222**, 143–255.

Miller, W. S. & Carlton, E. P. (1895). The relation of the cortex of the cat's kidney to the volume of the kidney, and an estimation of the number of glomeruli. *Trans. Wis. Acad. Sci. Arts Lett.* **10**, 525–38.

Moran, P. A. P. (1966). Measuring the length of a curve. *Biometrika* **53**, 359–64.

Moran, P. A. P. (1968). Statistical theory of a high-speed photoelectric planimeter. *Biometrika* **55**, 419–22.

Pache, J.-C., Roberts, N., Vock, P., Zimmermann, A. & Cruz-Orive, L. M. (1993). Vertical LM sectioning and CT scanning designs for stereology: application to human lung. *J. Microsc.* **170**, 9–24.

Rosiwal, A. (1898). Über geometrische Gesteinsanalysen. Ein einfacher Weg zur ziffermässigen Feststellung des Quantitätsverhältnisses der Mineralbestandteile gemengter Steine. *Verhandlungen der Kaiserlich-Königlichen Geologischen Reichsanstalt Wien*, 143–75.

Royall, R. M. (1970). On finite population sampling theory under certain linear regression models. *Biometrika* **57**, 377–87.

Steinhaus, H. (1929). Sur la portée pratique et théorique de quelques théorèmes sur la mesure de ensembles de droites. In *Comptes Rendues 1er Congr. Mathémathiciens des Pays Slaves*, pp. 348–54. Warszawa, Poland: Ksiaznica Atlas.

Steinhaus, H. (1954). Length, shape and area. *Colloq. Math.* **3**, 1–13.

Sterio, D. C. (1984). The unbiased estimation of number and size of arbitrary particles using the disecor. *J. Microsc.* **134**, 127–36.

Stoyan, D., Kendall, W. S. & Mecke, J. (1995). *Stochastic Geometry and Its Applications*, 2nd ed. Chichester, UK: John Wiley.

Sweet, R. A., Dorph-Petersen, K.-A. & Lewis, D. A. (2005). Mapping auditory core, lateral belt, and parabelt cortices in the human superior temporal gyrus. *J. Comp. Neurol.* **491**, 270–89.

Tang, Y. & Nyengaard, J. R. (2004). Length estimation of nerve fibers in human white matter using isotropic uniformly random sections. In *Quantitative Methods in Neuroscience—A Neuroanatomical Approach*, Ed. S. M. Evans, A. M. Janson & J. R. Nyengaard, pp. 249–63. New York: Oxford University Press.

Thompson, E. (1930). Quantitative microscopic analysis. *J. Geol.* **38**, 193–222.

Thompson, W. R. (1932). The geometric properties of microscopic configurations. I. General aspects of projectometry. *Biometrika* **24**, 21–26.

Thompson, W. R., Hussey, R., Matteis, J. T., Meredith, W. C., Wilson, G. C. & Tracy, F. E. (1932). The geometric properties of microscopic configurations. II. Incidence and volume of islands of Langerhans in the pancreas of a monkey. *Biometrika* **24**, 27–38.

Weibel, E. R. (1978). The non-statistical nature of biological structure and its implications on sampling for stereology. In *Geometrical Probability and Biological Structures: Buffon's 200th Anniversary*, Ed. R. E. Miles & J. Serra. Lecture Notes in Biomathematics 23. Berlin: Springer.

Weil, W. & Schneider, R. (2008). *Stochastic and Integral Geometry*. Heidelberg: Springer.

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